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On nonlinear scalarization methods in set-valued optimization

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Abstract: Based on the relationship between two sets with respect to a convex cone, we introduce six different solution concepts on set-valued optimization problems. By using a nonlinear scalarization method, we obtain optimal sufficient conditions for efficient solutions of set-valued optimization problems.

Key words: Nonlinear scalarization, vector optimization, set-valued optimization, set-valued maps, optimality conditions.

1 Introduction

In recent study on set-valued optimization problems, some solution concepts are defined by the efficiency of vectors as elements of set-valued objective functions based on a preorder which is a comparison between vectors with respect to a convex cone; see, [4] and [6]. In this paper, based on the comparisons between two sets introduced in [2], we introduce six different solution concepts on the same problem but by defining six types of efficiency on images of set-valued objective functions directly. By using a nonlinear scalarization method involving $h_C(y;k) := \inf\{t : y \in tk - C\}$ where $C \neq Y$ is a convex cone with nonempty interior in a real topological vector space Y and $k \in \operatorname{int} C$, we obtain optimal sufficient conditions for efficient solutions of set-valued optimization problems.

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2 Relationships Between Two Sets

In this section, we introduce relationships between two sets in a vector space. Throughout this section, let Z be a real ordered topological vector space with the vector ordering $\leq_C$ induced by a convex cone C : for $x, y \in Z$,

$$x \leq_C y \text{ if } y - x \in C.$$

First, we consider comparisons between two vectors. There are two types of comparable cases and in-comparable case. Comparable cases are as follows: for $a, b \in Z$,

$$(1)\ a \in b - C \text{ (i.e., } a \leq_C b), \qquad (2)\ a \in b + C \text{ (i.e., } b \leq_C a).$$

When we replace a vector $b \in Z$ with a set $B \subset Z$, that is, we consider comparison between a vector and a set, there are four types of comparable cases and in-comparable case. Comparable cases are as follows: for $a \in Z, B \subset Z$,

$$(1)\ A \subset (b - C), \qquad (2)\ A \cap (b - C) \neq \phi,$$
$$(3)\ A \cap (b + C) \neq \phi, \qquad (4)\ A \subset (b + C).$$

By the same way, when we replace a vector $a \in Z$ with a set $A \subset Z$, that is, we consider comparison between two sets with respect to C, there are twelve types of some what comparable cases and in-comparable case. For two sets $A, B \subset Z$, A would be inferior to B if we have one of the following situations:

$$(1)\ A \subset (\cap_{b \in B}(b - C)), \qquad (2)\ A \cap (\cap_{b \in B}(b - C) \neq \phi,$$
$$(3)\ (\cup_{a \in A}(a + C)) \supset B, \qquad (4)\ (\cup_{a \in A}(a + C)) \cup B,$$
$$(5)\ (\cap_{a \in A}(a + C)) \supset B, \qquad (6)\ ((\cap_{a \in A}(a + C)) \cap B) \neq \phi,$$
$$(7)\ A \subset (\cup_{b \in B}(b - C)), \qquad (8)\ (A \cap (\cup_{b \in B}(b - C)) \neq \phi).$$

Also, there are eight converse situations in which B would be inferior to A. Actually relationships (1) and (4) coincide with relationships (5) and (8), respectively. Therefore, we define the following six kinds of classification for set-relationships.

Definition 2.1 (Set-relationships in [2]) Given nonempty sets $A, B \subset Z$, we define six types of relationships between A and B as follows:

$$(1)\ A \leq_C^{(1)} B \text{ by } A \subset \cap_{b \in B}(b - C), \qquad (2)\ A \leq_C^{(2)} B \text{ by } A \cap (\cap_{b \in B}(b - C)) \neq \phi,$$
$$(3)\ A \leq_C^{(3)} B \text{ by } \cup_{a \in A}(a + C) \supset B, \qquad (4)\ A \leq_C^{(4)} B \text{ by } (\cap_{a \in A}(a + C)) \cap B \neq \phi,$$
$$(5)\ A \leq_C^{(5)} B \text{ by } A \subset \cup_{b \in B}(b - C), \qquad (6)\ A \leq_C^{(6)} B \text{ by } A \cap (\cup_{b \in B}(b - C)) \neq \phi.$$

Proposition 2.1 *For nonempty sets $A, B \in Z$ and a convex cone C in Z, the following statements hold:*

$$A \leq_C^{(1)} B \text{ implies } A \leq_C^{(2)} B; \qquad A \leq_C^{(1)} B \text{ implies } A \leq_C^{(4)} B;$$
$$A \leq_C^{(2)} B \text{ implies } A \leq_C^{(3)} B; \qquad A \leq_C^{(4)} B \text{ implies } A \leq_C^{(5)} B;$$
$$A \leq_C^{(3)} B \text{ implies } A \leq_C^{(6)} B; \qquad A \leq_C^{(5)} B \text{ implies } A \leq_C^{(6)} B.$$

3 Nonlinear Scalarization

At first, we introduce a nonlinear scalarization for set-valued maps and show some properties on a characteristic function and scalarizing functions introduced in this section.

Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. We assume that $C \neq Y$, which is equivalent to

$$\operatorname{int} C \cap (-\operatorname{cl} C) = \emptyset \tag{3.1}$$

for a convex cone with nonempty interior, where $\operatorname{int} C$ and $\operatorname{cl} C$ denote the interior and the closure of C, respectively.

To begin with, we define a characteristic function

$$h_C(y;k) := \inf\{t : y \in tk - C\}$$

where $k \in \operatorname{int} C$ and moreover $-h_C(-y;k) = \sup\{t : y \in tk + C\}$. This function $h_C(y;k)$ has been treated in some papers; see, [5] and [1], and it is regarded as a generalization of the Tchebyshev scalarization. Essentially, $h_C(y;k)$ is equivalent to the smallest strictly monotonic function with respect to $\operatorname{int} C$ defined by Luc in [3]. Note that $h_C(\cdot;k)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$, and hence it is sublinear and continuous.

Now, we give some useful properties of this function h_C.

Lemma 3.1 *Let $y \in Y$, then the following statements hold:*

(i) *If $y \in -\operatorname{int} C$, then $h_C(y;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $h_C(y;k) < 0$, then $y \in -\operatorname{int} C$.*

Proof. First we prove the statement (i). Suppose that $y \in -\operatorname{int} C$, then there exists an absorbing neighborhood V_0 of 0 in Y such that $y + V_0 \subset -\operatorname{int} C$. Since V_0 is absorbing, for all $k \in \operatorname{int} C$, there exists $t_0 > 0$ such that $t_0 k \in V_0$. Therefore, $y + t_0 k \in y + V_0 \subset -\operatorname{int} C$. Hence, we have

$$\inf\{t : y \in tk - C\} \leq -t_0 < 0,$$

which shows that $h_C(y;k) < 0$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \operatorname{int} C$ such that $h_C(y;k) < 0$. Then, there exist $t_0 > 0$ and $c_0 \in C$ such that $y = -t_0 k - c_0 = -(t_0 k + c_0)$. Since $t_0 k \in \operatorname{int} C$ and C is a convex cone, we have $y \in -\operatorname{int} C$. ∎

Remark 3.1 By combining statements (i) and (ii) above, we have the following: there exists $k \in \operatorname{int} C$ such that $h_C(y;k) < 0$ if and only if $y \in -\operatorname{int} C$.

Lemma 3.2 *Let $y \in Y$, then the following statements hold:*

(i) *If* $y \in -\mathrm{cl}\, C$, *then* $h_C(y;k) \leq 0$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If there exists* $k \in \mathrm{int}\, C$ *with* $h_C(y;k) \leq 0$, *then* $y \in -\mathrm{cl}\, C$.

Proof. First we prove the statement (i). Suppose that $y \in -\mathrm{cl}\, C$. Then, there exist a net $\{y_\lambda\} \subset -C$ such that y_λ converges to y. For each y_λ, since $y_\lambda \in 0 \cdot k - C$ for all $k \in \mathrm{int}\, C$, $h_C(y_\lambda;k) \leq 0$ for all $k \in \mathrm{int}\, C$. By the continuity of $h_C(\cdot;k)$, $h_C(y;k) \leq 0$ for all $k \in \mathrm{int}\, C$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \mathrm{int}\, C$ such that $h_C(y;k) \leq 0$. In the case $h_C(y;k) < 0$, from (ii) of Lemma 3.1, it is clear that $y \in -\mathrm{cl}\, C$. Then we assume that $h_C(y;k) = 0$ and show that $y \in -\mathrm{cl}\, C$. By the definition of h_C, for each $n = 1, 2, \ldots$, there exists $t_n \in R$ such that

$$h_C(y;k) \leq t_n < h_C(y;k) + \frac{1}{n} \tag{3.2}$$

and

$$y \in t_n k - C. \tag{3.3}$$

From condition (3.2), $\lim_{n\to\infty} t_n = 0$. From condition (3.3), there exists $c_n \in C$ such that $y = t_n k - c_n$, that is, $c_n = t_n k - y$. Since $c_n \to -y$ as $n \to \infty$, we have $y \in -\mathrm{cl}\, C$. ∎

Remark 3.2 By combining statements (i) and (ii) above, we have the following: there exists $k \in \mathrm{int}\, C$ such that $h_C(y;k) \leq 0$ if and only if $y \in -\mathrm{cl}\, C$.

Lemma 3.3 *Let* $y \in Y$, *then the following statements hold:*

(i) *If* $y \in \mathrm{int}\, C$, *then* $h_C(y;k) > 0$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If* $y \in \mathrm{cl}\, C$, *then* $h_C(y;k) \geq 0$ *for all* $k \in \mathrm{int}\, C$.

The following lemma shows (strictly) monotone property on $h_C(\cdot;k)$.

Lemma 3.4 *Let* $y, \bar{y} \in Y$, *then the following statements hold:*

(i) *If* $y \in \bar{y} + \mathrm{int}\, C$, *then* $h_C(y;k) > h_C(\bar{y};k)$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If* $y \in \bar{y} + \mathrm{cl}\, C$, *then* $h_C(y;k) \geq h_C(\bar{y};k)$ *for all* $k \in \mathrm{int}\, C$.

Lemma 3.5 *Let* $y, \bar{y} \in Y$ *and* $k \in \mathrm{int}\, C$, *then the following statements hold:*

(i) *If* $h_C(y;k) > h_C(\bar{y};k)$, *then* $h_C(y - \bar{y};k) > 0$;

(ii) *If* $h_C(y;k) \geq h_C(\bar{y};k)$, *then* $h_C(y - \bar{y};k) \geq 0$.

Remark 3.3 In the above lemma, we note that each converse does not hold.

Now, we consider several characterizations for images of a set-valued map by the nonlinear and strictly monotone characteristic function h_C. We observe the following four types of scalarizing functions:

$$\begin{aligned}
&(1) \quad \psi_C^F(x;k) := \sup\{h_C(y;k) : y \in F(x)\},\\
&(2) \quad \varphi_C^F(x;k) := \inf\{h_C(y;k) : y \in F(x)\},\\
&(3) \quad -\varphi_C^{-F}(x;k) = \sup\{-h_C(-y;k) : y \in F(x)\},\\
&(4) \quad -\psi_C^{-F}(x;k) = \inf\{-h_C(-y;k) : y \in F(x)\}.
\end{aligned}$$

Functions (1) and (4) have symmetric properties and then results for function (4) $-\psi_C^{-F}$ can be easily proved by those for function (1) ψ_C^F. Similarly, the results for function (3) $-\varphi_C^{-F}$ can be deduced by those for function (2) φ_C^F. By using these four functions we measure each image of set-valued map F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image with respect to convex cone C.

Proposition 3.1 *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \cap (-\operatorname{int} C) \neq \emptyset$, then $\varphi_C^F(x;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $\varphi_C^F(x;k) < 0$, then $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.*

Proof. Let $x \in X$ be given. First we prove the statement (i). Suppose that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$. Then, there exists $y \in F(x) \cap (-\operatorname{int} C)$. By (i) of Lemma 3.1, for all $k \in \operatorname{int} C$, $h_C(y;k) < 0$, and hence, $\varphi_C^F(x;k) < 0$.

Next we prove the statement (ii). Suppose that there exists $k \in \operatorname{int} C$ such that $\varphi_C^F(x;k) < 0$. Then, there exist $\varepsilon_0 > 0$ and $y_0 \in F(x)$ such that

$$h_C(y_0;k) \leq \inf_{y \in F(x)} h_C(y;k) + \varepsilon_0 < 0.$$

By (ii) of Lemma 3.1, we have $y_0 \in -\operatorname{int} C$, which implies that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$. ∎

Remark 3.4 By combining statements (i) and (ii) above, we have the following: there exists $k \in \operatorname{int} C$ such that $\varphi_C^F(x;k) < 0$ if and only if $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.

Proposition 3.2 *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \subset -\operatorname{int} C$ and $F(x)$ is a compact set, then $\psi_C^F(x;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $\psi_C^F(x;k) < 0$, then $F(x) \subset -\operatorname{int} C$.*

Proof. Let $x \in X$ be given. First we prove the statement (i). Assume that $F(x)$ is a compact set and suppose that $F(x) \subset -\operatorname{int} C$. Then, for all $k \in \operatorname{int} C$,

$$F(x) \subset \bigcup_{t>0} (-tk - \operatorname{int} C).$$

By the compactness of $F(x)$, there exist $t_1, \ldots, t_m > 0$ such that

$$F(x) \subset \bigcup_{i=1}^{m} (-t_i k - \operatorname{int} C).$$

Since $-t_q k - \operatorname{int} C \subset -t_p k - \operatorname{int} C$ for $t_p < t_q$, there exists $t_0 := \min\{t_1, \ldots, t_m\} > 0$ such that $F(x) \subset -t_0 k - \operatorname{int} C$. For each $y \in F(x)$, we have

$$h_C(y; k) = \inf\{t : y \in tk - C\} \leq -t_0.$$

Hence,

$$\psi_C^F(x; k) = \sup_{y \in F(x)} h_C(y; k) \leq -t_0 < 0.$$

Next, we prove the statement (ii). Suppose that there exists $k \in \operatorname{int} C$ such that $\psi_C^F(x; k) < 0$. Then, for all $y \in F(x)$, $h_C(y; k) < 0$. By (ii) of Lemma 3.1, we have $y \in -\operatorname{int} C$, and hence $F(x) \subset -\operatorname{int} C$. ∎

Remark 3.5 By combining statements (i) and (ii) above, we have the following: there exists $k \in \operatorname{int} C$ such that $\psi_C^F(x; k) < 0$ if and only if $F(x) \subset -\operatorname{int} C$. When we replace $F(x)$ in (i) of Proposition 3.2 by $\operatorname{cl} F(x)$, the assertion still remains.

Moreover, we can replace (i) in Proposition 3.2 by another relaxed form.

Corollary 3.1 *Let $x \in X$ and assume that there exists a compact set B such that $B \subset -\operatorname{int} C$. If $F(x) \subset B - C$, then $\psi_C^F(x; k) < 0$ for all $k \in \operatorname{int} C$.*

Proof. Let $x \in X$, and assume that there exists a compact set B such that $B \subset -\operatorname{int} C$ and $F(x) \subset B - C$. By applying (i) of Proposition 3.2 to B instead of $F(x)$, for all $k \in \operatorname{int} C$,

$$\sup_{y \in B} h_C(y; k) < 0.$$

Since $F(x) \subset B - C$, it follows from (i) of Lemma 3.1 and the subadditivity of $h_C(\cdot; k)$ that

$$h_C(y; k) \leq \sup_{z \in B} h_C(z; k)$$

for each $y \in F(x)$. Therefore, $\psi_C^F(x; k) < 0$ for all $k \in \operatorname{int} C$. ∎

Proposition 3.3 *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$, then $\varphi_C^F(x; k) \leq 0$ for all $k \in \operatorname{int} C$;*

(ii) *If $F(x)$ is a compact set and there exists $k \in \operatorname{int} C$ with $\varphi_C^F(x; k) \leq 0$, then $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$.*

Proof. Let $x \in X$ and we prove the statement (i). Suppose that $F(x) \cap (-\mathrm{cl}\, C) \neq \emptyset$. Then, there exists $y \in F(x) \cap (-\mathrm{cl}\, C)$. By (i) of Lemma 3.2, for all $k \in \mathrm{int}\, C$, $h_C(y;k) \leq 0$, and hence $\varphi_C^F(x;k) \leq 0$.

Next, we prove the statement (ii). Suppose that there exists $k \in \mathrm{int}\, C$ such that $\varphi_C^F(x;k) \leq 0$. In the case $\varphi_C^F(x;k) < 0$, from (ii) of Proposition 3.1, it is clear that $F(x) \cap (-\mathrm{cl}\, C) \neq \emptyset$. So we assume that $\varphi_C^F(x;k) = 0$ and show that $F(x) \cap (-\mathrm{cl}\, C) \neq \emptyset$. By the definition of φ_C^F, for each $n = 1, 2, \ldots$, there exist $t_n \in R$ and $y_n \in F(x)$ such that $y_n \in t_n k - C$ and

$$\varphi_C^F(x;k) \leq t_n < \varphi_C^F(x;k) + \frac{1}{n}. \tag{3.4}$$

From (3.4), $\lim_{n \to \infty} t_n = 0$. Since $F(x)$ is compact, we may suppose that $y_n \to y_0$ for some $y_0 \in F(x)$ without loss of generality (taking subsequence). Therefore, $y_n - t_n k \to y_0$ and then $y_0 \in -\mathrm{cl}\, C$, which shows that $F(x) \cap (-\mathrm{cl}\, C) \neq \emptyset$. ∎

Remark 3.6 By combining statements (i) and (ii) above, we have the following: under the compactness of $F(x)$, there exists $k \in \mathrm{int}\, C$ such that $\varphi_C^F(x;k) \leq 0$ if and only if $F(x) \cap (-\mathrm{cl}\, C) \neq \emptyset$. Otherwise, there are counter-examples violating the statement (ii) such as an unbounded set approaching $-\mathrm{cl}\, C$ asymptotically or an open set whose boundary intersects $-\mathrm{cl}\, C$.

Proposition 3.4 *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \subset -\mathrm{cl}\, C$, then $\psi_C^F(x;k) \leq 0$ for all $k \in \mathrm{int}\, C$;*

(ii) *If there exists $k \in \mathrm{int}\, C$ with $\psi_C^F(x;k) \leq 0$, then $F(x) \subset -\mathrm{cl}\, C$.*

Proof. Let $x \in X$ be given. First we prove the statement (i). Suppose that $F(x) \subset -\mathrm{cl}\, C$. Then, for each $y \in F(x)$, it follows from (i) of Lemma 3.2 that $h_C(y;k) \leq 0$ for all $k \in \mathrm{int}\, C$, and hence $\psi_C^F(x;k) \leq 0$ for all $k \in \mathrm{int}\, C$.

Next, we prove the statement (ii). Suppose that there exists $k \in \mathrm{int}\, C$ such that $\psi_C^F(x;k) \leq 0$. Then, for all $y \in F(x)$, $h_C(y;k) \leq 0$. By (ii) of Lemma 3.2, we have $y \in -\mathrm{cl}\, C$, and hence $F(x) \subset -\mathrm{cl}\, C$. ∎

Remark 3.7 By combining statements (i) and (ii) above, we have the following: there exists $k \in \mathrm{int}\, C$ such that $\psi_C^F(x;k) \leq 0$ if and only if $F(x) \subset -\mathrm{cl}\, C$.

4 Optimality Conditions

In this section, we introduce new definitions of efficient solution for set-valued optimization problems. Using the sclarization method introduced in Section 3, we obtain optimal sufficient conditions on such efficiency. Throughout this section, let X be a nonempty set, Y a real ordered topological vector space with convex cone C. We assume that $C \neq Y$ and $\mathrm{int}\, C \neq \emptyset$. Let $F : X \to 2^Y$ be a set-valued map. A set-valued optimization problem is written as

(SVOP) min F(x) subject to $x \in V$, where $V = \{x \in X : F(x) \neq \phi\}$.

In this problem, we were defined an efficient solution as follows ever. Vector $x_0 \in V$ is an efficient solution of (SVOP) if there exists $y_0 \in F(x_0)$ such that $F(x)\backslash\{y_0\} \cap (y_0 - C) = \phi$ for all $x \in V$. This type of solution is defined based on a comparison between vectors. However F is a set-valued map, so it is natural to define efficient solution concepts based on direct comparisons between sets given in Definition 2.1.

Definition 4.1 (Efficient solution of (SVOP)) $x_0 \in V$ is said to be an efficient (resp. weakly efficient) solution for (SVOP) with respect to $\leq_C^{(i)}$for $i = 1, \dots, 6$ if there exists no $x \in V\backslash\{x_0\}$ satisfying $F(x) \leq_C^{(i)} F(x_0)$ (resp. $F(x) \leq_{\mathrm{int}\, C}^{(i)} F(x_0)$) for $i = 1, \dots, 6$, respectively.

Using sclarization functions introduced in Section 3, we obtain the following optimal sufficient conditions for (SVOP).

Theorem 4.1 *Let $x_0 \in V$. If there exists $k \in \mathrm{int}C$ such that either $\varphi_C^F(x_0; k) \leq \psi_C^F(x; k)$ or $-\psi_C^{-F}(x_0; k) \leq -\varphi_C^{-F}(x; k)$ for any $x \in V$, then x_0 is a weakly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{int}\, C}^{(1)}$.*

Proof. Suppose that there exists $k \in \mathrm{int}C$ such that either $\varphi_C^F(x_0; k) \leq \psi_C^F(x; k)$ or $-\psi_C^{-F}(x_0; k) \leq -\varphi_C^{-F}(x; k)$ for any $x \in V$. Assume that x_0 is not a weakly efficient solution with respect to $\leq_{\mathrm{int}\, C}^{(1)}$. Then there exist $\bar{x} \in V$ such that $F(\bar{x}) \leq_{\mathrm{int}\, C}^{(1)} F(x_0)$ (that is, $\bar{y} \in \cap_{y_0 \in F(x_0)}(y_0 - \mathrm{int}\, C)$ for any $\bar{y} \in F(\bar{x})$). From condition (i) in Lemma 3.4, it follows that for any $k \in \mathrm{int}\, C$, $h_C(\bar{y}; k) < h_C(y_0; k)$ and $-h_C(-\bar{y}; k) < -h_C(-y_0; k)$ for $\bar{y}$ and y_0 satisfying with $\bar{y} \in F(\bar{x})$ and $y_0 \in F(x_0)$. Hence we get $\psi_C^F(\bar{x}; k) < \varphi_C^F(x_0; k)$ and $-\varphi_C^{-F}(\bar{x}; k) < -\psi_C^{-F}(x_0; k)$, which are contradictions to the assumption. ∎

Theorem 4.2 *Let $x_0 \in V$.If there exist $k \in \mathrm{int}C$ such that either $\varphi_C^F(x_0; k) \leq \varphi_C^F(x; k)$ or $-\psi_C^{-F}(x_0; k) \leq -\psi_C^{-F}(x; k)$ for any $x \in V$, then x_0 is a weakly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{int}\, C}^{(2)}$.*

Theorem 4.3 *Let $x_0 \in V$.If there exist $k \in \mathrm{int}C$ such that either $\varphi_C^F(x_0; k) \leq \varphi_C^F(x; k)$ or $-\psi_C^{-F}(x_0; k) \leq -\psi_C^{-F}(x; k)$ for any $x \in V\backslash\{x_0\}$, then x_0 is a weakly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{int}\, C}^{(3)}$.*

Theorem 4.4 *Let $x_0 \in V$.If there exist $k \in \mathrm{int}C$ such that either $\psi_C^F(x_0; k) \leq \psi_C^F(x; k)$ or $-\varphi_C^{-F}(x_0; k) \leq -\varphi_C^{-F}(x; k)$ for any $x \in V$, then x_0 is a weakly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{int}\, C}^{(4)}$.*

Theorem 4.5 *Let $x_0 \in V$.If there exist $k \in \mathrm{int}C$ such that either $\psi_C^F(x_0; k) \leq \psi_C^F(x; k)$ or $-\varphi_C^{-F}(x_0; k) \leq -\varphi_C^{-F}(x; k)$ for any $x \in V\backslash\{x_0\}$, then x_0 is a weakly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{int}\, C}^{(5)}$.*

Theorem 4.6 *Let* $x_0 \in V$*.If there exist* $k \in \mathrm{int}C$ *such that either* $\psi_C^F(x_0;k) \leq \varphi_C^F(x;k)$ *or* $-\varphi_C^{-F}(x_0;k) \leq -\psi_C^{-F}(x;k)$ *for any* $x \in V\backslash\{x_0\}$*, then* x_0 *is a weakly efficient solution for* (SVOP) *with respect to* $\leq^{(6)}_{\mathrm{int}\,C}$.

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